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Simplicity of the heart of a nondegenerate Jordan system ☆

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Dedicated to Professor Kevin McCrimmon on the occasion of his 60th birthday

Abstract

In this paper we prove that the heart of a nondegenerate Jordan system (algebra, triple system or pair) is either simple or zero. We also obtain Herstein type results relating the hearts of associative systems and those of their corresponding Jordan systems. © 2001 Elsevier Science B.V. All rights reserved.

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The notion of heart (i.e., the intersection of all nonzero ideals) appears naturally when studying associative or Jordan systems. Indeed, having a nonzero heart can be considered as a strong version of primeness for semiprime or nondegenerate systems. As an example, in the proof of the Jordan algebra version of Kaplansky's Theorem [10, 1.2], having or not having a nonzero heart plays a major role. In the recent results on local and subquotient inheritance of simplicity [6], having a "sufficiently big" heart comes out as the natural (weaker) substitute of simplicity. The heart is used in a similar way in the Herstein type results of [7] relating Jordan and associative ideals. There is a common feature in all of these different particular situations: under suitable conditions, the heart of a nondegenerate Jordan (resp. semiprime associative) system is shown to be simple when it is nonzero.

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Indeed, using Andrunakievich's Lemma, it is not hard to prove that the heart of an arbitrary semiprime associative system is either zero or simple. However, the Jordan version of Andrunakievich's Lemma is false even for linear Jordan algebras (cf. [18]) and thus the study of the simplicity of the heart in Jordan systems requires different techniques. For nondegenerate linear Jordan algebras the simplicity of the heart when it is nonzero is a consequence of the following stronger result by Medvedev [18, p. 933] and Skosyrskii [21, Corollary 3.1]: every minimal ideal of a linear Jordan algebra is either simple or trivial. The techniques are mainly combinatorial and strongly dependent on the linearity, i.e., on the existence of $\frac{1}{2}$ in the ring of scalars. Prior to the results by Medvedev and Skosyrskii, Nam and McCrimmon [20] give a first approach to the study of minimal ideals in quadratic Jordan algebras. In this more general setting, they show that minimal ideals in arbitrary (quadratic) Jordan algebras are either trivial or $\mathscr{D}$-simple, i.e., not having proper ideals invariant under all derivations. As a consequence, the heart of an arbitrary Jordan algebra is either trivial or $\mathscr{D}$-simple.

In this paper we extend the above mentioned results on the simplicity of the heart to arbitrary nondegenerate Jordan systems (algebras, triple systems and pairs) without any restriction on the ring of scalars, though we do not attack the more general problem of the simplicity of minimal ideals. We will base our proofs on the structure theory of strongly prime Jordan systems [2,3,17,22], the use of hermitian or Zelmanov polynomials [1,17] and the recent Herstein type theorems [7] relating ideals of associative systems and their corresponding Jordan systems.

After a preliminary section, we study in Section 1 ideals of hermitian polynomials, namely of hearty eaters, both in the algebra and triple system settings. Indeed we just elaborate on the constructions given in [1,17] to show the existence of ideals of arbitrarily voracious hearty eaters. These are used in the following two sections to obtain the main results in the paper. In the second section we show that the heart of a nondegenerate Jordan algebra is either zero or simple, and in the third section we give triple system and pair analogues of that result.

Many consequences can be obtained from the above results. Some of them are collected in the last section of the paper, where we also obtain Herstein type results relating the heart of an associative system R (resp. the $*$-heart of an associative system with involution $*$) and the heart of its symmetrization $R^{(+)}$ (resp. any ample subspace $H_0(R,*)$ of $*$-symmetric elements of R).

0. Preliminaries

0.1. We will deal with associative and Jordan algebras, pairs and triple systems over an arbitrary ring of scalars Φ. The reader is referred to [2,11,12,17] for basic results, notation and terminology, though we will stress some definitions and basic notions. The identities JPx listed in [12] will be quoted with their original numbering without explicit reference to [12].

- Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

- For a Jordan pair $V=(V^+,V^-)$ we will denote the products by $Q_x y$, for any $x\in V^\sigma$, $y\in V^{-\sigma}$, $\sigma=\pm$, with linearizations denoted by $Q_{x,z}y=\{x,y,z\}=D_{x,y}z$.
- A Jordan triple system T is given by its products $P_x y$, for any $x,y\in T$, with linearizations denoted by $P_{x,z}y=\{x,y,z\}=L_{x,y}z$.

0.2. One can obtain Jordan systems from associative systems by symmetrization: If R is an associative algebra, we can obtain a Jordan algebra denoted by $R^{(+)}$, over the same Φ-module, with products built out of the associative product by $x^2=xx$, $U_x y=xyx$, for any $x,y\in R$. Similarly, a Jordan pair (resp. Jordan triple system) $R^{(+)}$ can be obtained from an associative pair $R=(R^+,R^-)$ (resp. an associative triple system R) by defining $Q_x y=xyx$, for any $x\in R^\sigma$, $y\in R^{-\sigma}$, $\sigma=\pm$ (resp. $P_x y=xyx$, for any $x,y\in R$).

A Jordan system (algebra, pair or triple system) is said to be *special* if it is a subsystem of $R^{(+)}$ for some associative system R.

0.3. A particularly important example of special Jordan systems are ample subspaces or subpairs of associative systems with involution:

- If R is an associative algebra with involution $*$, a Φ-submodule $H_0(R,*)$ contained in the set of symmetric elements $H(R,*)$ is said to be an *ample subspace* of R if it contains all traces and norms of the elements of R ($\{x\}=x+x^*, xx^*\in H_0(R,*)$ for any $x\in R$) and $xH_0(R,*)x^*\subseteq H_0(R,*)$ for any $x\in R$ [13, p. 387; 17, 0.8′].
- If $R=(R^+,R^-)$ is an associative pair with polarized involution $*$, an *ample subpair* $H_0(R,*)=(H_0^+,H_0^-)$ is a pair of submodules of symmetric elements ($H_0^\sigma\subseteq H(R^\sigma,*)$) containing all traces $\{x\}=x+x^*$ of the elements of R^σ and satisfying $xH_0^{-\sigma}x^*\subseteq H_0^\sigma$, for any $x\in R^\sigma$, $\sigma=\pm$. To obtain the notion of *ample subspace* of an associative triple system R with involution, simply forget the superscripts in the case of pairs [2, pp. 209–210; 8, 1.7].

Clearly, $H_0(R,*)$ is a subsystem of $R^{(+)}$ in all three cases.

When dealing with pairs, involutions will be assumed to be of polarized type.

0.4. We will deal with the usual notions of regularity in Jordan systems: *simplicity*, *primeness*, *nondegeneracy*, whose definitions and basic properties can be found in the general references listed above. Concerning nondegeneracy we will stress a well known fact that will be used several times in the sequel:

If J is a nondegenerate Jordan system (algebra, triple system or pair) then every nonzero ideal of J is nondegenerate (cf. [15, 3.5]).

0.5. A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P=U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T)=(T,T)$ with products $Q_x y=P_x y$, for any $x,y\in T$. From a Jordan pair $V=(V^+,V^-)$ one can get a (*polarized*) Jordan triple system $T(V)=V^+\oplus V^-$ by defining $P_{x^+\oplus x^-}(y^+\oplus y^-)=Q_{x^+}y^-\oplus Q_{x^-}y^+$ [12, 1.13, 1.14].

Similarly, one can consider the underlying triple system of an associative algebra, as well as functors $V(\;)$ and $T(\;)$ between the categories of associative pairs and associative triple systems.

0.6. *Regularity conditions through the functor* $T(\,)$ [2, p. 230]: Let V be a Jordan (resp. associative) pair.

(i) V is nondegenerate (resp. semiprime) if and only if $T(V)$ is nondegenerate (resp. semiprime),

(ii) V is strongly prime (resp. prime) if and only if $T(V)$ is strongly prime (resp. prime),

(iii) V is simple if and only if $T(V)$ is simple.

If R is an associative system with involution $*$, primeness can be replaced by $*$-primeness in (ii) [5, 1.4(i)], and simplicity can be replaced by $*$-simplicity in (iii).

0.7. *Homotopes of Jordan and associative systems*:

- Given an associative pair $R=(R^+,R^-)$ and $a\in R^{-\sigma}$, the Φ-module R^σ becomes an associative algebra, denoted $R^{\sigma(a)}$ and called the *a-homotope of* R, with product

$$x\cdot_a y = xay,$$

for any $x,y\in R^\sigma$.

- Given a Jordan pair $V=(V^+,V^-)$ and $a\in V^{-\sigma}$, the Φ-module V^σ becomes a Jordan algebra, denoted $V^{\sigma(a)}$ and called the *a-homotope of* V, with products

$$x^{(2,a)}=Q_x a,\quad U_x^{(a)}y=Q_xQ_a y$$

for any $x,y\in V^\sigma$.

- Homotopes of a Jordan or associative algebra or triple system J at an element a are simply those of the Jordan pair $V(J)$.

The above notions are compatible with the functors $V(\,)$ and $T(\,)$ (cf. [5, 0.5]).

0.8. The *heart* Heart(S) of an associative or Jordan system S is the intersection of all nonzero ideals of S. For an associative system S with involution $*$, the *$*$-heart* $*$-Heart(S) of S is the intersection of all nonzero $*$-ideals of S.

0.9. Assume that J is a strongly prime Jordan system and L is an ideal of J such that L is simple. Then, for any nonzero ideal I of J, $I\cap L\neq 0$ by strong primeness and it is an ideal of L. Hence $L=I\cap L$ by simplicity of L and $L\subseteq I$. This argument applies with suitable changes to associative systems with and without involution, so that we have:

(i) Let J be a strongly prime Jordan (resp. prime associative) algebra, triple system or pair. If L is an ideal of J which is a simple system, then $L=\mathrm{Heart}(J)$.

(ii) Let R be a $*$-prime associative algebra, triple system or pair. If L is a $*$-ideal which is a $*$-simple system, then $L=*\text{-}\mathrm{Heart}(R)$.

0.10. For a Jordan algebra J, the *centroid* $\Gamma(J)$ of J is defined as [14, p. 298]

$$\Gamma(J)=\{\gamma\in\mathrm{End}_\Phi(J)\mid \gamma U_x=U_x\gamma, U_{\gamma x}=\gamma^2U_x, (\gamma x)^2=\gamma^2x^2,\\ V_x\gamma=\gamma V_x,\ \text{for all } x\in J\}.$$

By [14, Theorem 2], for a Jordan algebra J with zero extreme radical (for example, if J is semiprime), $\Gamma(J)$ is an associative commutative ring and J can be considered as a Jordan algebra over $\Gamma(J)$. When J is prime, $\Gamma(J)$ is a domain acting on J without torsion [17, 4.2; 16, 2.8] and one can consider the so-called *central closure* $\Gamma(J)^{-1}J$ of J, which is an algebra over the field of fractions $\Gamma(J)^{-1}\Gamma(J)$.

Indeed one has a Jordan version of Posner–Rowen Theorem for associative algebras:

0.11. Theorem (Anquela et al. [10, 1.1]).
Let J be a strongly prime, PI Jordan algebra. Then $\Gamma(J)^{-1}J$ is simple and unital.

A triple system version of the above results is still not known (see [19, 6.1, 6.2]). However, a weak version of it is obtained in [19] by means of the following substitutes of the usual centroid and the usual central closure.

0.12. Let J be a Jordan triple system and I be an ideal of J. A linear mapping $f: I \to J$ is called a *J-homomorphism* if, for all $y \in I$, $x,z \in J$, it satisfies:
(i) $f(P_x y) = P_x f(y)$,
(ii) $f(P_I J) \subseteq I$ and $f^2(P_y x) = P_{f(y)}x$,
(iii) $f(\{y,x,z\}) = \{f(y),x,z\}$.
The set of all J-homomorphisms defined on I is denoted by $\mathrm{Hom}_J(I,J)$ [19, 1.1].

0.13. A pair (f,I), where $f \in \mathrm{Hom}_J(I,J)$ is said to be *permissible* if the ideal I of J is essential (it hits every nonzero ideal of J) [19, 1.1].

0.14. In the set of all permissible maps (f,I) of a Jordan triple system J the relation "$\sim$" is defined by

$(f,I) \sim (g,L)$ if there is an essential ideal K of J, contained in $I \cap L$, such that $f(x) = g(x)$, for all $x \in K$,

and turns out to be an equivalence relation. The quotient set $\mathscr{C}(J)$ is called the *extended centroid* of J. The equivalence class of a permissible map (f,I) will be denoted by $[f,I]$ [19, 1.4].

For a nondegenerate Jordan triple system J, there is a natural way to define a Φ-algebra structure on $\mathscr{C}(J)$ [19, 1.11, 1.13]. Indeed:
(i) If J is nondegenerate, then $\mathscr{C}(J)$ is a commutative, associative, unital, von Neumann regular Φ-algebra [19, 1.15].
(ii) If J is strongly prime, then $\mathscr{C}(J)$ is a field [19, 1.15].

0.15. Let J be a nondegenerate Jordan triple system. In the (free) scalar extension $\mathscr{C}(J) \otimes_\Phi J$ the set

$$R = \left\{ \sum_i (\rho_i \lambda_i \otimes x_i - \rho_i \otimes f_i(x_i) \mid \lambda_i, \rho_i \in \mathscr{C}(J), (f_i, I_i) \in \lambda_i,\ x_i \in I_i \right\}$$

is an ideal of $\mathscr{C}(J) \otimes_\Phi J$ [19, 3.2]. In [19, 3.6], an ideal $\tilde{M}$ is found in the quotient $\tilde{J} = (\mathscr{C}(J) \otimes_\Phi J)/R$ so that, up to isomorphism, J is still contained in $\tilde{J}/\tilde{M}$ and this is tight over J. This latter quotient $\tilde{J}/\tilde{M}$, denoted $\mathscr{C}(J)J$, is called the *extended central closure* of J. Rather than the explicit construction of $\tilde{M}$ and $\mathscr{C}(J)$, we will need the following properties of the central closure:

(i) $\mathscr{C}(J)J$ is a Jordan triple system over $\mathscr{C}(J)$ which is a tight extension of J. Thus, we can assume that $J \subseteq \mathscr{C}(J)J$, rather than explicitly using the injective map $x \mapsto ((1 \otimes x) + R) + \tilde{M}$ [19, 3.5, 3.6].

(ii) If $x \in I$ and $\lambda = [f, I] \in \mathscr{C}(J)$ the product λx in $\mathscr{C}(J)J$ is given by $\lambda x = f(x)$ (see the definition of R).

When dealing with Jordan triple systems we will use the following weak version of Posner–Rowen Theorem:

0.16. Theorem (Montaner [19, 6.1]).

Let J be a strongly prime, homotope-PI Jordan triple system. Then $\mathscr{C}(J)J$ is simple.

1. Hearty-eaters in Jordan systems

In [1] and [17] nonzero ideals of hearty pentad eaters for Jordan triple systems and algebras are found. By taking suitable powers of those ideals, we will find nonzero ideals of hearty n-tad eaters for all n:

1.1. Recall the notion of *adic family* on a special Jordan algebra [17, 13.4–6]. The subset of all *hearty n-tad eaters* [17, 13.8] in the *free special Jordan algebra* $\mathrm{FSJ}_{alg}(X)$ (X is an infinite set of variables) will be denoted $\mathscr{HE}_n(X)$. A hearty n-tad eater is a Jordan polynomial $p = p(x_1, \ldots, x_r)$ which eats adic families in $\mathrm{FSJ}_{alg}(X)$:

(i) $F_n(y_1, \ldots, y_{n-1}, p) = \sum F_3(q_1, q_2, q_3)$,

where $q_i = q_i(x_1, \ldots, x_r, y_1, \ldots, y_{n-1})$, $i = 1, 2, 3$, lie in $\mathrm{FSJ}_{alg}(X)$. Replacing variables by the unit element shows that a hearty n-tad eater p eats in lower levels of adic families:

(ii) If $m \leqslant n$, then $F_m(y_1, \ldots, y_{m-1}, p) = \sum F_3(q_1, q_2, q_3)$.

Following [17, 13.8], we denote by $\mathscr{H}_n(X)$ the core of $\mathscr{HE}_n(X)$, i.e., the biggest ideal of $\mathrm{FSJ}_{alg}(X)$ consisting of hearty n-tad eaters. Using (ii), we obtain

(iii) $\mathscr{HE}_{n+1}(X) \subseteq \mathscr{HE}_n(X)$, hence $\mathscr{H}_{n+1}(X) \subseteq \mathscr{H}_n(X)$ for all n.

Notice that, by arguing as in [1, 3.14(3)], but using the fact that $\mathscr{H}_n(X)$ is an ideal, the elements in $\mathscr{H}_n(X)$ eat adic m-tads from any position:

(iv) If $p \in \mathscr{H}_n(X)$ and $m \leqslant n$, then

$$F_m(y_1, \ldots, y_k, p, y_{k+1}, \ldots, y_{m-1}) = \sum F_3(q_1, q_2, q_3),$$

for any $0 \leqslant k \leqslant m - 1$.

In [4, 3.6], hearty n-tad eaters are shown to be exactly those polynomials which eat associative n-tads. Indeed, we just need this weaker property of hearty eaters:

(v) If $p \in \mathscr{H}_n(X)$ and $m \leqslant n$, then

$$y_1 \ldots y_k \, p \, y_{k+1} \ldots y_{m-1} = \sum q_1 q_2 q_3,$$

for any $0 \leqslant k \leqslant m - 1$.

1.2. Similarly, *adic families* on a special Jordan triple system are defined in [1, 3.7]. Unlike on Jordan algebras, adic families on special Jordan triple systems can only occur at odd levels, and we do not have the possibility of reducing the level by introducing unit elements. Thus, the definition of hearty eaters in triple systems is changed so that the triple version of (1.1)(ii) holds. We will match the algebra notation unlike [1] and denote by $\mathscr{H}ET_n(X)$ (n is odd) the subset of all *hearty n-tad eaters* [1, 3.12] in the *free special Jordan triple system* $\mathrm{FSJ}_{trip}(X)$ (X is an infinite set of variables). A hearty n-tad eater (n is odd) in the triple sense is a Jordan polynomial $p=p(x_1,\dots,x_r)$ which eats adic families in $\mathrm{FSJ}_{trip}(X)$ at all odd levels no greater than n:

(i) If m is odd and $m \leqslant n$, then

$$F_m(y_1,\dots,y_{m-1},p)=\sum F_3(q_1,q_2,q_3)$$

and

$$F_m(y_1,\dots,y_{m-2},p,y_{m-1})=\sum F_3(q_4,q_5,q_6),$$

where $q_i=q_i(x_1,\dots,x_r,y_1,\dots,y_{n-1})$, $i=1,\dots,6$, lie in $\mathrm{FSJ}_{trip}(X)$.

We denote by $\mathscr{H}T_n(X)$ the core of $\mathscr{H}ET_n(X)$, i.e., the biggest ideal of $\mathrm{FSJ}_{trip}(X)$ consisting of hearty n-tad eaters. Clearly,

(ii) $\mathscr{H}ET_{n+2}(X)\subseteq \mathscr{H}ET_n(X)$, hence $\mathscr{H}T_{n+2}(X)\subseteq \mathscr{H}T_n(X)$ for all odd n.

Arguing as in the algebra case, we have that the elements in $\mathscr{H}T_n(X)$ eat adic m-tads from any position:

(iii) If $p\in \mathscr{H}T_n(X)$, m is odd and $m\leqslant n$, then

$$F_m(y_1,\dots,y_k,p,y_{k+1},\dots,y_{m-1})=\sum F_3(q_1,q_2,q_3),$$

for any $0\leqslant k\leqslant m-1$.

As for algebras we just need the weaker associative n-tad eating property of hearty eaters:

(iv) If $p\in \mathscr{H}T_n(X)$, m is odd and $m\leqslant n$, then

$$y_1\dots y_k\,p\,y_{k+1}\dots y_{m-1}=\sum q_1q_2q_3,$$

for any $0\leqslant k\leqslant m-1$.

We will show that nonzero hearty n-tad eaters exist in the core $\mathscr{H}_n(X)$ of $\mathscr{H}E_n(X)$ (resp. $\mathscr{H}T_n(X)$ of $\mathscr{H}ET_n(X)$) for arbitrarily big n.

1.3. Proposition. *For all $n\geqslant 5$, $\mathscr{H}_n(X)$ is a nonzero linearization invariant ideal of* $\mathrm{FSJ}_{alg}(X)$. *Moreover, $\mathscr{H}_n(X)$ contains nonzero Clifford polynomial identities.*

Proof. In [17, 13.10] it is shown that $\mathscr{H}_n(X)$ is linearization invariant for any $n\geqslant 4$.

We claim that, for all $n\geqslant 5$,

$$U_{\mathscr{H}_n(X)}\mathscr{H}_n(X)\subseteq \mathscr{H}_{n+2}(X). \tag{1}$$

To show (1), since $U_{\mathscr{H}_n(X)}\mathscr{H}_n(X)$ is an ideal of $\mathrm{FSJ}_{alg}(X)$, we just need to show that $U_{\mathscr{H}_n(X)}\mathscr{H}_n(X) \subseteq \mathscr{H}E_{n+2}(X)$: Let $p, q \in \mathscr{H}_n(X)$; for any adic family $\{F_k\}_{k\in\mathbf{N}}$,

$$\begin{aligned}
F_{n+2}(y_1,\dots,y_{n+1},U_p q) &= F_{n+4}(y_1,\dots,y_{n+1},p,q,p) \text{ (by [17, 13.4 (AIII)])}\\
&= \sum F_{n+2}(y_1,\dots,y_{n-3},q_1,q_2,q_3,q,p)\\
&\quad \text{(since } p\in\mathscr{H}_n(X)\subseteq\mathscr{H}_5(X) \text{ by (1.1)(iii))}\\
&= \sum F_n(y_1,\dots,y_{n-4},q_4,q_5,q_6,p) \text{ (since } q\in\mathscr{H}_n(X)\subseteq\mathscr{H}_5(X) \text{ by (1.1)(iii))}\\
&= \sum F_3(q_7,q_8,q_9)
\end{aligned}$$

since $p\in\mathscr{H}_n(X)$.

On the other hand, in [17, 14.2; 1, p. 182], an element $p\in\mathscr{H}_5(X)$ is found which takes the value $e_{12}+e_{21}$ in the Jordan algebra of symmetric 3×3 matrices $H_3(\Phi)$ over Φ, under a suitable substitution of the variables. This readily implies that $\mathscr{H}_5(H_3(\Phi)) = H_3(\Phi)$, hence $\mathscr{H}_n(H_3(\Phi)) = H_3(\Phi)$ for all $n \geqslant 5$, using (1), (1.1)(iii) and the fact that $U_{H_3(\Phi)}H_3(\Phi) = H_3(\Phi)$. In particular, $\mathscr{H}_n(X)$ contains nonzero Clifford polynomial identities, for all $n\geqslant 5$. $\square$

1.4. Proposition. *For all odd $n \geqslant 5$, $\mathscr{H}T_n(X)$ is a nonzero linearization invariant ideal of* $\mathrm{FSJ}_{trip}(X)$. *Moreover, $\mathscr{H}T_n(X)$ contains nonzero Clifford homotope polynomial identities.*

Proof. By [1, 3.14(1)] for all odd $n\geqslant 5$, $\mathscr{H}T_n(X)$ is linearization invariant. Moreover, $\mathscr{H}T_5(X) = \mathscr{H}ET_5(X)$ by [1, 3.16].

In [5, 0.8] it is shown that, for any odd n, $\mathscr{H}ET_{n+4}(X)$ contains the semiideal $\mathscr{H}T_5(X)^n$, where the powers $\mathscr{H}T_5(X)^n$ are defined inductively by

$$\mathscr{H}T_5(X)^1 = \mathscr{H}T_5(X), \quad \mathscr{H}T_5(X)^n = P_{\mathscr{H}T_5(X)}(\mathscr{H}T_5(X)^{n-2}).$$

Notice that:

(1) the elements in $\mathscr{H}T_5(X)^n$ eat adic m-tads for any odd $m\leqslant n+4$ from any position [5, 0.7].

Let $I_n(X)$ be the ideal of $\mathrm{FSJ}_{trip}(X)$ generated by $\mathscr{H}T_5(X)^n$, i.e.,

$$I_n(X) = \mathscr{H}T_5(X)^n + P_{\mathrm{FSJ}_{trip}(X)}(\mathscr{H}T_5(X)^n).$$

We claim that $I_n(X) \subseteq \mathscr{H}ET_{n+2}(X)$, hence $I_n(X) \subseteq \mathscr{H}T_{n+2}(X)$: $\mathscr{H}T_5(X)^n \subseteq \mathscr{H}ET_{n+4}(X) \subseteq \mathscr{H}ET_{n+2}(X)$ by (1.2)(ii), hence one just need to prove that an element $P_q p$, where $p\in\mathscr{H}T_5(X)^n$ and $q\in\mathrm{FSJ}_{trip}(X)$, eats adic m-tads for any odd $m\leqslant n+2$; but indeed $P_q p$ eats adic m-tads for any odd $m\leqslant n+2$ from any position, by using [1, 3.6(A3)], since p eats adic m-tads for any odd $m\leqslant n+4$ from any position, by (1).

By [1, 4.5; 3, Section 6], there exists an element $p\in\mathscr{H}T_5(X)$ which is a homotope polynomial, i.e., there exists $q=q(x_1,\dots,x_r)\in\mathrm{FSJ}_{alg}(X)$ such that $p=p(x_1,\dots,x_r,y)= q(y;x_1,\dots,x_r)$ is the evaluation of q on the homotope algebra $\mathrm{FSJ}_{trip}(X)^{(y)}$ of $\mathrm{FSJ}_{trip}(X)$. Moreover, a suitable evaluation of q (i.e., of p, where y takes the value $1 = e_{11} + e_{22} + e_{33}$) on $H_3(\Phi)$ yields $e_{23}+e_{32}$. This shows that p is a nonzero Clifford homotope polynomial identity inside $\mathscr{H}T_5(X)$. Now, for any positive integer n, let $q_n = q^n$, $p_n(x_1,\dots,x_r,y) = q_n(y;x_1,\dots,x_r)$. Clearly

$$\begin{aligned} p_{4n+1}(x_1,\dots,x_r,y) &= (U^{(y)}_{q(y;x_1,\dots,x_r)})^{2n} q(y;x_1,\dots,x_r) \\ &= (P_{q(y;x_1,\dots,x_r)}P_y)^{2n} q(y;x_1,\dots,x_r) = (P_pP_y)^{2n}p \\ &= (P_pP_yP_pP_y)^n p \\ &= (P_pP_{P_yp})^n p \in \mathscr{H}T_5(X)^{4n+1} \end{aligned}$$

since $p, P_yp \in \mathscr{H}T_5(X)$. Thus $p_{4n+1}(x_1,\dots,x_r,y) \in I_{4n+1}(X) \subseteq \mathscr{H}T_{4n+3}(X) \subseteq \mathscr{H}T_n(X)$ by (1.2)(ii) for all odd $n \geqslant 5$. Notice that p_{4n+1} is a Clifford homotope identity since the same replacement of variables of q_{4n+1} in $H_3(\Phi)$ as the one mentioned above for q yields $(e_{23} + e_{32})^{4n+1} = e_{23} + e_{32} \neq 0$. $\square$

2. The heart of a nondegenerate Jordan algebra

2.1. It is well known the fact that the heart (resp. the $*$-heart) of a semiprime associative algebra (resp. a semiprime associative algebra with involution $*$) R is simple (resp. $*$-simple) when it is nonzero: If $0 \neq I$ is the heart (resp. the $*$-heart) of R, then it is not trivial since R is semiprime. Moreover, for any nonzero ideal (resp. $*$-ideal) L of I the ideal $\tilde{L}$ of R generated by L (which is a $*$-ideal in the case with involution) satisfies $\tilde{L}^3 \subseteq L$ (Andrunakievich's Lemma). Since $0 \neq L \subseteq \tilde{L}$, $\tilde{L}^3 \neq 0$ by semiprimeness of R and $I \subseteq \tilde{L}^3 \subseteq L$, i.e., $I = L$, which shows that I is simple (resp. $*$-simple). Our main task in this section will be to prove an analogue for Jordan algebras of the above result.

We begin with a remark dealing with the cube of the heart of a nondegenerate Jordan algebra.

2.2. Remark. For every nondegenerate Jordan algebra J,

$$\mathrm{Heart}(J)^3 := U_{\mathrm{Heart}(J)}\mathrm{Heart}(J) = \mathrm{Heart}(J).$$

Indeed, we can assume that $I = \mathrm{Heart}(J)$ is nonzero since, otherwise, our assertion is obvious. Notice that $I^3 = U_I I$ is an ideal of J which is nonzero since I is nondegenerate by (0.4). Hence $I \subseteq I^3$ and I^3 is clearly contained in I, which yields $I = I^3$.

The next lemma will allow us to use the structure theory of strongly prime Jordan algebras when studying the heart.

2.3. Lemma. *If J is a nondegenerate Jordan algebra and* $\mathrm{Heart}(J) \neq 0$ *then J is prime* (*hence strongly prime*).

Proof. Let I, L be nonzero ideals of J. We have that $U_{\mathrm{Heart}(J)}\mathrm{Heart}(J) \subseteq U_I L$ and $U_{\mathrm{Heart}(J)}\mathrm{Heart}(J) \neq 0$ since $\mathrm{Heart}(J)$ is nondegenerate by (0.4). Thus $U_I L \neq 0$. $\square$

2.4. Remark. The proof of (2.3) can be easily adapted to the associative case to prove the following assertion: If R is a semiprime associative algebra (resp. a semiprime associative algebra with involution $*$) and $\mathrm{Heart}(R) \neq 0$ (resp. $*$-$\mathrm{Heart}(R) \neq 0$), then R is prime (resp. $*$-prime).

When a nondegenerate algebra is PI, having a nonzero heart has further consequences than the mere simplicity of the heart. This is shown in the next result, which can be considered as a part of our main theorem for algebras.

2.5. Proposition. *If J is a nondegenerate, PI Jordan algebra and the heart* $\mathrm{Heart}(J)$ *of J is nonzero, then* $\mathrm{Heart}(J) = J$ *and J is simple and unital.*

Proof. Let $I := \mathrm{Heart}(J) \neq 0$.

By (2.3), J is strongly prime. Since J is PI, then, by (0.11), the central closure $\Gamma(J)^{-1}J$ of J is simple and unital.

Let $0 \neq \gamma \in \Gamma(J)$. We claim that $\gamma I \subseteq I$: for any $x, y \in I$, $\gamma(U_x y) = U_x \gamma y \in U_I J \subseteq I$, which shows $\gamma I^3 \subseteq I$; but $I^3 = I$ by (2.2). Moreover, γI is an ideal of J: for any $x \in I$, $y \in J$,

$$U_{\gamma x} y = \gamma^2 U_x y = \gamma(U_x(\gamma y)) \in \gamma(U_I J) \subseteq \gamma I,$$

$$(\gamma x)^2 = \gamma^2 x^2 = \gamma(\gamma x^2) \in \gamma(\gamma I) \subseteq \gamma I,$$

$$(\gamma x) \circ y = \gamma(x \circ y) \in \gamma(I \circ J) \subseteq \gamma I,$$

$$U_y(\gamma x) = \gamma(U_y x) \in \gamma(U_J I) \subseteq \gamma I.$$

Using that $\Gamma(J)$ acts on J without torsion, γI is nonzero, which implies $I \subseteq \gamma I$, hence $I = \gamma I$.

Since we have $\Gamma(J)I = I$, $\Gamma(J)^{-1}I$ makes sense and is an ideal of $\Gamma(J)^{-1}J$. Moreover, $0 \neq I \subseteq \Gamma(J)^{-1}I$, hence $\Gamma(J)^{-1}I = \Gamma(J)^{-1}J$ by simplicity of $\Gamma(J)^{-1}J$.

We claim that $\Gamma(J)^{-1}I = I$. Indeed, for any $0 \neq \gamma \in \Gamma(J)$ and $x \in I$, since $\gamma I = I$, there exists $z \in I$ such that $x = \gamma z$. But this implies $\gamma^{-1}x = z \in I$.

Thus $\Gamma(J)^{-1}J = \Gamma(J)^{-1}I = I \subseteq J \subseteq \Gamma(J)^{-1}J$, which implies $I = J = \Gamma(J)^{-1}J$ which is simple and unital. $\square$

Now, we can prove our main result for Jordan algebras.

2.6. Theorem. *If J is a nondegenerate Jordan algebra then the heart* $\mathrm{Heart}(J)$ *of J is either simple or zero.*

Proof. Let us assume that $I := \mathrm{Heart}(J) \neq 0$.

If J is PI, then the theorem follows from (2.5). Thus we may assume that J is not PI.

By (2.3), J is strongly prime hence J is special by [17, 15.2] since it is not PI. Thus we can take an associative $*$-tight envelope R of J, i.e., $J \subseteq H(R, *)$, R is generated as an associative algebra by J and every nonzero $*$-ideal of R hits J.

Let $\tilde{I}$ be the ideal of R generated by I. Notice that $\tilde{I}$ is a nonzero $*$-ideal of R.

(I) $\tilde{I} = *$-Heart(R): For any nonzero $*$-ideal L of R, $L \cap J$ is a nonzero ideal of J by $*$-tightness, hence $I \subseteq L \cap J \subseteq L$ and $\tilde{I} \subseteq L$. Thus $\tilde{I} \subseteq *$-Heart(R), but $*$-Heart$(R) \subseteq \tilde{I}$ since $\tilde{I}$ is a nonzero $*$-ideal of R.

(II) $\tilde{I}$ is $*$-simple by (2.1) since R is semiprime by $*$-tightness.

(III) $IJ + JI \subseteq II$: Let $a \in I$, $x \in J$. By (2.2), $a = \sum U_{a_i} b_i$, where $a_i, b_i \in I$. Hence

$$ax = \left(\sum U_{a_i} b_i\right) x = \left(\sum a_i b_i a_i\right) x = \sum (a_i\{b_i, a_i, x\} - a_i x a_i b_i)$$
$$= \sum (a_i\{b_i, a_i, x\} - (U_{a_i} x) b_i) \in I\{I, I, J\} + (U_I J) I \subseteq II.$$

Similarly, $xa \in II$.

(IV) $\tilde{I} = I + II + III + \cdots$: Indeed, $\tilde{I} = I + RI + IR + RIR \subseteq I + II + III + \cdots$ by (III), since R is generated by J.

(V) $I \subseteq \mathscr{H}_n(J)$ for any $n \geqslant 4$: Since J is not PI, by (1.3) and (1.1)(iii), $\mathscr{H}_n(J)$ is a nonzero ideal of J, hence $I \subseteq \mathscr{H}_n(J)$.

(VI) I is an ample subspace of symmetric elements of $\tilde{I}$, so that we can write $I = H_0(\tilde{I}, *)$: Since clearly $I \subseteq H(\tilde{I}, *)$, we only have to prove that I contains all traces and norms of elements in $\tilde{I}$ and also xIx^* for any $x \in \tilde{I}$:

- For any $x_1, \ldots, x_n \in I$, $n \geqslant 4$,

$$\begin{aligned} x_1 \ldots x_n + (x_1 \ldots x_n)^* = \{x_1 \ldots x_n\} &\in \{\mathscr{H}_n(J) I \overbrace{J \ldots J}^{n-2}\} \quad \text{(by (V))} \\ &\subseteq \sum \{q_1(I, J, \ldots, J), q_2(I, J, \ldots, J), q_3(I, J, \ldots, J)\} \\ &\subseteq \{I, J, J\} + \{J, I, J\} + \{J, J, I\} \subseteq I \end{aligned}$$

  since by homogeneity in (1.1)(v), in every summand, at least one the factors q_1, q_2, q_3 contains a variable evaluated in I which appears in all of its Jordan monomials. The above, with (IV) implies that I contains all traces of elements in $\tilde{I}$.
- Using (IV), every norm of an element in $\tilde{I}$ is a sum of traces of elements in $\tilde{I}$ and elements of the form $x_1 \ldots x_n x_n \ldots x_1$, where $x_1, \ldots, x_n \in I$. Since $x_1 \ldots x_n x_n \ldots x_1 = U_{x_1} \ldots U_{x_{n-1}} x_n^2 \in U_I \ldots U_I I^2 \subseteq I$, every norm of an element in $\tilde{I}$ lies in I.
- By (IV) again, if $x \in \tilde{I}$, the elements in xIx^* are sums of traces of elements in $\tilde{I}$ and elements of the form $x_1 \ldots x_n y x_n \ldots x_1$, where $x_1, \ldots, x_n, y \in I$. Since $x_1 \ldots x_n y x_n \ldots x_1 = U_{x_1} \ldots U_{x_n} y \in U_I \ldots U_I I \subseteq I$, we have $xIx^* \subseteq I$ for any $x \in \tilde{I}$.

(VII) Finally, I is simple by [7, 2.7(ii)], using (II) and the fact that $I^3 = I$ by (2.2). □

3. The heart of a nondegenerate Jordan triple system or pair

We will obtain triple system versions of the results in the previous section.

3.1. As in the algebra case, the heart (resp. the $*$-heart) of a semiprime associative triple system (resp. a semiprime associative triple system with involution $*$) R

is simple (resp. $*$-simple) when it is nonzero. The proof [6, 4.5] follows by using an Andrunakievich-like argument which is obviously still valid in the case with involution.

The proof of (2.2) cannot be applied to triple systems since the cubes of ideals are not ideals anymore in the triple case (cf. [2, 2.9]). However, the triple version of (2.2) is still true.

3.2. Remark (Anquela and Cortés [6, 4.4]).
For every nondegenerate Jordan triple system J,

$$\mathrm{Heart}(J)^3 := P_{\mathrm{Heart}(J)}\mathrm{Heart}(J) = \mathrm{Heart}(J).$$

The proof of (2.3) can be extended to triple systems simply by replacing U's by P's.

3.3. Lemma. *If J is a nondegenerate Jordan triple system and* $\mathrm{Heart}(J) \neq 0$ *then J is prime* (*hence strongly prime*).

3.4. Remark. Similarly, if R is a semiprime associative triple system (resp. a semiprime associative triple system with involution $*$) and $\mathrm{Heart}(R) \neq 0$ (resp. $*$-$\mathrm{Heart}(R) \neq 0$), then R is prime (resp. $*$-prime).

To obtain the triple system version of (2.5), we will need to use extended central closures instead of usual central closures.

3.5. Proposition. *If J is a nondegenerate*, *homotope-PI Jordan triple system and the heart* $\mathrm{Heart}(J)$ *of J is nonzero*, *then* $\mathrm{Heart}(J) = J$ *and J is simple*.

Proof. Let $I := \mathrm{Heart}(J) \neq 0$.

By (3.3), J is strongly prime. Since J is homotope-PI, the extended central closure $\mathscr{C}(J)J$ of J is simple by (0.16).

Let $0 \neq \gamma \in \mathscr{C}(J)$. We claim that $\gamma I \subseteq I$: $\gamma = [f, L]$, where L is a nonzero ideal of J; hence $I \subseteq L$ and $\gamma x = f(x)$ for any $x \in I$ by (0.15)(ii). Moreover, since $I = I^3$ by (3.2), $x = \sum P_{a_i} b_i$, hence $f(x) = \sum P_{a_i} f(b_i) \in P_I J \subseteq I$, using (0.12)(i).

Now, we have that I is a nonzero ideal of $\mathscr{C}(J)J$, hence $I = \mathscr{C}(J)J$ by simplicity of $\mathscr{C}(J)J$.

Thus $J \subseteq \mathscr{C}(J)J = I \subseteq J$, and $J = I$ is a simple Jordan triple system. $\square$

Now, we can prove our main result for Jordan triple systems, which is just the triple system version of (2.6).

3.6. Theorem. *If J is a nondegenerate Jordan triple system then the heart* $\mathrm{Heart}(J)$ *of J is either simple or zero.*

Proof. Let us assume that $I := \mathrm{Heart}(J) \neq 0$.

If J is homotope-PI, then the theorem follows from (3.5). Thus we may assume that J is not homotope-PI.

By (3.3), J is strongly prime and hence J is special by [2, 4.1; 22, Theorem 5] since J is not homotope-PI. Thus we can take an associative $*$-tight envelope R of J, i.e., $J \subseteq H(R,*)$, R is generated as an associative triple system by J and every nonzero $*$-ideal of R hits J.

Let $\tilde{I}$ be the ideal of R generated by I. Notice that $\tilde{I}$ is a nonzero $*$-ideal of R.

(I) $\tilde{I} = *$-Heart(R): Formally, the proof of part (I) of the proof of (2.6) holds.

(II) $\tilde{I}$ is $*$-simple by (3.1) since R is semiprime by $*$-tightness.

(III) $IJJ + JJI + JIJ \subseteq III$: Let $a \in I$, $x, y \in J$. By (3.2), $a = \sum P_{a_i} b_i$, where $a_i, b_i \in I$. Hence

$$axy = \left(\sum P_{a_i} b_i\right) xy = \left(\sum a_i b_i a_i\right) xy = \sum (a_i\{b_i, a_i, x\}y - a_i x a_i b_i y)$$
$$= \sum (a_i\{b_i, a_i, x\}y - (P_{a_i}x) b_i y) \in I\{I, I, J\}y + (P_I J)Iy \subseteq IIy.$$

Also,

$$xay = x\left(\sum P_{a_i} b_i\right) y = \left(\sum x a_i b_i a_i\right) y = \sum(\{x, a_i, b_i\}a_i y - b_i a_i x a_i y)$$
$$= \sum(\{x, a_i, b_i\}a_i y - b_i(P_{a_i}x)y) \in \{J, I, I\}Iy + I(P_I J)y \subseteq IIy.$$

We have shown, for any $y \in J$,

$$IJy + JIy \subseteq IIy. \tag{2}$$

Similarly,

$$yJI + yIJ \subseteq yII. \tag{3}$$

Now,

$$\begin{aligned} IJJ &\subseteq IIJ \text{ (by (2))} \subseteq III \text{ (by (3))},\\ JJI &\subseteq JII \text{ (by (3))} \subseteq III \text{ (by (2))},\\ JIJ &\subseteq IIJ \text{ (by (2))} \subseteq III \text{ (by (3))}. \end{aligned}$$

(IV) $\tilde{I} = I + III + IIIII + \cdots$: Indeed we just need to notice that $I + III + IIIII + \cdots$ is an ideal of R, which follows from (III), since R is generated by J.

(V) $I \subseteq \mathscr{H}T_n(J)$ for any odd $n \geqslant 5$: Since J is not homotope-PI, $\mathscr{H}T_n(J)$ is a nonzero ideal of J by (1.4), hence $I \subseteq \mathscr{H}T_n(J)$.

(VI) I is an ample subspace of symmetric elements of $\tilde{I}$, so that we can write $I = H_0(\tilde{I}, *)$: Since clearly $I \subseteq H(\tilde{I}, *)$, we only have to prove that I contains all traces of elements in $\tilde{I}$ and also xIx^* for any $x \in \tilde{I}$:

- For any $x_1, \ldots, x_n \in I$, where n is odd and $n \geqslant 5$,

$$\begin{aligned} x_1 \ldots x_n + (x_1 \ldots x_n)^* = \{x_1 \ldots x_n\} &\in \{\mathscr{H}T_n(J) I \overbrace{J \ldots J}^{n-2}\} \text{ (by (V))}\\ &\subseteq \sum \{q_1(I, J, \ldots, J), q_2(I, J, \ldots, J), q_3(I, J, \ldots, J)\}\\ &\subseteq \{I, J, J\} + \{J, I, J\} + \{J, J, I\} \subseteq I \end{aligned}$$

since by homogeneity in (1.2)(iv), in every summand, at least one of the factors q_1, q_2, q_3 contains a variable evaluated in I which appears in all of its Jordan monomials. The above, with (IV) implies that I contains all traces of elements in $\tilde{I}$.

- By (IV) again, if $x \in \tilde{I}$, the elements in xIx^* are sums of traces of elements in $\tilde{I}$ and elements of the form $x_1 \dots x_n y x_n \dots x_1$, where $x_1, \dots, x_n, y \in I$. Since $x_1 \dots x_n y x_n \dots x_1 = P_{x_1} \dots P_{x_n} y \in P_I \dots P_I I \subseteq I$, we have $xIx^* \subseteq I$ for any $x \in \tilde{I}$.

(VII) Finally, I is simple by [7, 3.15(ii)], using (II) and the fact that $I^3 = I$ by (3.2). □

We will obtain pair analogues of the above results for triple systems by using the functor $T()$.

3.7. Let V be a Jordan (resp. associative) pair and $T(V)$ be the polarized triple system associated to V. Using the definition of the functor $T()$ immediately yields

(i) $T(V)^3 = T(V^3)$, where $V^3 = (Q_{V^+}V^-, Q_{V^-}V^+)$.

On the other hand, in [2, p. 230] it is shown that, for any semiprime Jordan pair (in particular, for any nondegenerate Jordan pair) V, every nonzero ideal of $T(V)$ contains a polarized ideal $T(I)$, where I is a nonzero ideal of V. The proof can be easily adapted to semiprime associative pairs with and without involution. This readily implies

(ii) If V is a semiprime Jordan or associative pair, then $\mathrm{Heart}(T(V)) = T(\mathrm{Heart}(V))$. If V is a semiprime associative pair with involution $*$, then $*$-$\mathrm{Heart}(T(V)) = T(*\text{-}\mathrm{Heart}(V))$.

Every homotope $V^{\sigma(a)}$, where $a \in V^{-\sigma}$, is clearly a subalgebra of the homotope $T(V)^{(a)}$ of $T(V)$. Conversely, the homotope $T(V)^{(a+b)}$, where $a \in V^+$, $b \in V^-$ is readily seen to be the direct sum $V^{+(b)} \boxplus V^{-(a)}$ of the homotope algebras $V^{+(b)}$, $V^{-(a)}$ of V. Thus

(iii) V is homotope-PI if and only if $T(V)$ is homotope-PI.

The above comments, together with (0.6), readily imply the pair versions of (3.1)–(3.6). As a consequence, we have:

3.8. Theorem. *If V is a nondegenerate Jordan pair then the heart* $\mathrm{Heart}(V)$ *of V is either simple or zero.*

4. Further results

The above results on the simplicity of the heart will allow us to relate the heart of a prime system and the heart of any of its subideals avoiding Andrunakievich's Lemma which is false for arbitrary Jordan systems (cf. [18]).

4.1. Corollary. (i) *Let J be a strongly prime Jordan (resp. prime associative) algebra, triple system or pair, and L be a nonzero subideal of J. Then* $\mathrm{Heart}(J) = \mathrm{Heart}(L)$.

(ii) *Let R be a $*$-prime associative algebra, triple system or pair with involution $*$ and L be a nonzero $*$-subideal of R. Then* $*$-$\mathrm{Heart}(R) = *$-$\mathrm{Heart}(L)$.

Proof. We will prove (i) for Jordan systems. The remaining cases of (i) and (ii) can be proved analogously.

Let J be a strongly prime Jordan system, L be a nonzero ideal of J and I be the heart of J and M be the heart of L. Notice that L is strongly prime by [15, 2.5].

For any nonzero ideal N of J, $N \cap L$ is a nonzero ideal of L by strong primeness of J. Hence $M \subseteq N \cap L$, hence

$$M \subseteq I. \tag{4}$$

If $I = 0$, then $M = 0$ by (4). If $I \neq 0$, then I is simple by (2.6), (3.6) or (3.8). Moreover, I is contained in L, which implies that I is a simple ideal of L. Since L is strongly prime, $I = M$ by (0.9).

We have shown

$$\mathrm{Heart}(J) = \mathrm{Heart}(L) \tag{5}$$

for any nonzero ideal L of J.

Since L is strongly prime, (5) applies to L to show that the heart of L coincide with the heart of any nonzero ideal of L. The argument can be iterated to obtain (5) for any subideal of J. □

The simplicity of the heart can also be used to study the heart through the functors (0.5) relating algebras, triple systems and pairs.

4.2. Corollary. (i) *Let J be a nondegenerate Jordan* (*resp. semiprime associative*) *algebra. Then the heart* $\mathrm{Heart}_{alg}(J)$ *of J is equal to the heart* $\mathrm{Heart}_{trip}(J)$ *of the underlying triple system of J.*

(ii) *Let R be an associative algebra with involution $*$. Then the $*$-heart* $*\text{-}\mathrm{Heart}_{alg}(R)$ *of R is equal to the $*$-heart* $*\text{-}\mathrm{Heart}_{trip}(R)$ *of the underlying triple system of R.*

Proof. We will prove (i) for Jordan algebras. The remaining case of (i) and (ii) can be proved analogously.

Let J be a nondegenerate Jordan algebra. Since every nonzero ideal of the algebra J is a nonzero ideal in the triple sense, we have

$$\mathrm{Heart}_{trip}(J) \subseteq \mathrm{Heart}_{alg}(J). \tag{6}$$

If $\mathrm{Heart}_{alg}(J) = 0$, then $\mathrm{Heart}_{trip}(J) = 0$ by (1).

If $\mathrm{Heart}_{alg}(J) \neq 0$, then it is simple by (2.6) and J is a strongly prime as an algebra by (2.3). We have that J is strongly prime as a triple system by [8, 1.13, 2.9] and $\mathrm{Heart}_{alg}(J)$ is an ideal of J in the triple sense which is simple as a triple system by [6, 3.4]. By (0.9)(i), $\mathrm{Heart}_{alg}(J) = \mathrm{Heart}_{trip}(J)$. □

4.3. Remark. The nice behaviour of the functor $T()$ with respect to the heart (3.7)(ii) is not shared by the functor $V()$, which does not preserve the heart as shown by the following example: Let W be a simple Jordan pair and $J = T(W) = W^+ \oplus W^-$. By

(0.6)(iii), J is simple, hence $J = \mathrm{Heart}(J)$, but $V(J)$ is a direct sum of two ideals,

$$V(J) = (W^+ \oplus 0, 0 \oplus W^-) \boxplus (0 \oplus W^-, W^+ \oplus 0),$$

which implies $\mathrm{Heart}(V(J)) = 0$.

However, when applied to algebras, the functor V preserves the heart.

4.4. Corollary. (i) *If J is a nondegenerate Jordan* (*resp. semiprime associative*) *algebra*, *then*

$$\mathrm{Heart}(V(J)) = V(\mathrm{Heart}(J)).$$

(ii) *If R is a semiprime associative algebra with involution $*$*, *then*

$$*\text{-}\mathrm{Heart}(V(R)) = V(*\text{-}\mathrm{Heart}(R)).$$

Proof. We will prove (i) for Jordan algebras. The remaining case of (i) and (ii) can be proved analogously.

Let J be a nondegenerate Jordan algebra. Since any nonzero ideal I of J gives rise to a nonzero ideal $V(I)$ of $V(J)$,

$$\mathrm{Heart}(V(J)) \subseteq V(\mathrm{Heart}(J)). \tag{7}$$

If $\mathrm{Heart}(J) = 0$, then $\mathrm{Heart}(V(J)) = 0$ by (7).

If $\mathrm{Heart}(J) \neq 0$, then $\mathrm{Heart}(J)$ is a simple algebra (2.6), and $V(\mathrm{Heart}(J))$ is a simple Jordan pair by [6, 3.4]. Moreover, $V(\mathrm{Heart}(J))$ is an ideal of $V(J)$, which is strongly prime by [8, 1.12] since J is strongly prime by (2.3). By (0.9)(i), $\mathrm{Heart}(V(J)) = V(\mathrm{Heart}(J))$. $\square$

4.5. Remark. Nondegeneracy is necessary in (4.2) and (4.4) as the following examples show:

(1) Let $J = R^{(+)}$, where R is the associative commutative algebra generated over a field Φ by an element a of order 3: $R = \Phi a \oplus \Phi a^2$, $a^3 = 0$. Obviously, J is trivial as a triple system, hence every subspace is an ideal and $\mathrm{Heart}_{trip}(J) = 0$. However, $\mathrm{Heart}_{alg}(J) = \Phi a^2$.

(2) Let J be a trivial one-dimensional algebra over a field Φ. Then $\mathrm{Heart}(J) = J$ since J is the only nonzero ideal of J. However, $V(J)$ is the direct sum of ideals

$$V(J) = (J, 0) \boxplus (0, J),$$

which implies that $\mathrm{Heart}(V(J)) = 0$.

By using Herstein's Theorems [7,13], we can also study the interaction of the heart with symmetrizations and ample subspaces.

4.6. Corollary. *Let R be a semiprime associative algebra*, *triple system or pair. Then* $\mathrm{Heart}(R^{(+)}) = \mathrm{Heart}(R)$.

Proof. Since every ideal of R is an ideal of $R^{(+)}$, we have $\mathrm{Heart}(R^{(+)}) \subseteq \mathrm{Heart}(R)$. By [7, 1.5, 1.6; 13, Corollary of p. 384], every nonzero ideal of $R^{(+)}$ contains a nonzero ideal of R, which implies $\mathrm{Heart}(R^{(+)}) \supseteq \mathrm{Heart}(R)$. $\square$

Following [7, 2.1], if R is an associative algebra with involution $*$, $H_0 = H_0(R,*)$ is an ample subspace of symmetric elements of R, and B is a $*$-ideal of R, we define

$$K(B,H_0) = \left\{ b + b^* + \sum_i \lambda_i b_i b_i^* + \sum_j b_j h_j b_j^* \mid b, b_i, b_j \in B, h_j \in H_0, \lambda_i \in \Phi \right\}.$$

Similarly, $KP(B^\sigma, H_0)$, $\sigma = \pm$ (resp., $KT(B,H_0)$) is introduced in [7, 3.1] (resp., [7, 3.9]) when R is an associative pair (resp., triple system) with involution $*$.

4.7. Corollary. *Let R be an associative algebra, triple system or pair with involution $*$, $H_0(R,*)$ be an ample subspace of symmetric elements of R, and $I = *$-*Heart(R). Then*

(i) *if R is a semiprime algebra, then* Heart$(H_0(R,*)) = K(I, H_0(R,*))$,

(ii) *if R is a $*$-prime triple system, then* Heart$(H_0(R,*))$ *is the ideal of $H_0(R,*)$ generated by $KT(I,H_0(R,*))$, i.e.,*

$$\mathrm{Heart}(H_0(R,*)) = KT(I,H_0(R,*)) + P_{H_0(R,*)} KT(I,H_0(R,*)),$$

(iii) *if R is a $*$-prime pair, then* Heart$(H_0(R,*))$ *is the ideal of $H_0(R,*)$ generated by $(KP(I^+,H_0(R,*)), KP(I^-,H_0(R,*)))$, i.e.,*

$$\mathrm{Heart}(H_0(R,*))^\sigma = KP(I^\sigma, H_0(R,*)) + Q_{H_0(R^\sigma,*)} KP(I^{-\sigma}, H_0(R,*)), \quad \sigma = \pm.$$

In all cases Heart$(H_0(R,*))$ *is an ample subspace $H_0(I,*)$ of I and*

$$\mathrm{Heart}(H_0(R,*)) = 0 \iff *\text{-Heart}(R) = 0.$$

Proof. (i) Let L be a nonzero ideal of $H_0(R,*)$. By [7, 2.6], there exists a $*$-ideal L_1 of R such that $0 \neq K(L_1, H_0(R,*)) \subseteq L$. In particular, $L_1 \neq 0$, hence $I \subseteq L_1$ and $K(I,H_0(R,*)) \subseteq L$. We have shown that

$$\mathrm{Heart}(H_0(R,*)) \supseteq K(I,H_0(R,*)). \tag{8}$$

If Heart$(H_0(R,*)) = 0$, then $K(I,H_0(R,*)) = 0$ by (8), and (i) trivially holds.

Let us assume that Heart$(H_0(R,*)) \neq 0$. Let $\mathscr{I}$ be the set of nonzero $*$-ideals of R. By [7, 2.2, 2.5], for any $A \in \mathscr{I}$, $K(A,H_0(R,*))$ is a nonzero ideal of $H_0(R,*)$ contained in A. Thus

$$\mathrm{Heart}(H_0(R,*)) \subseteq \bigcap_{A \in \mathscr{I}} K(A,H_0(R,*)) \subseteq \bigcap_{A \in \mathscr{I}} A = I,$$

which shows that $I \neq 0$. Now $K(I,H_0(R,*))$ is a nonzero ideal of $H_0(R,*)$ by [7, 2.2, 2.5], and (8) readily implies (i).

(ii) If $H_0(R,*) = 0$, everything is obvious, so we can assume that $H_0(R,*) \neq 0$. Let L be a nonzero ideal of $H_0(R,*)$. By [7, 3.14], there exists a nonzero $*$-ideal L_1 of R such that $KT(L_1,H_0(R,*)) \subseteq L$, hence $I \subseteq L_1$ and $KT(I,H_0(R,*)) \subseteq L$. We have shown that

$$\mathrm{Heart}(H_0(R,*)) \supseteq KT(I,H_0(R,*)),$$

which implies

$$\mathrm{Heart}(H_0(R,*)) \supseteq KT(I,H_0(R,*)) + P_{H_0(R,*)}KT(I,H_0(R,*)). \tag{9}$$

If $\mathrm{Heart}(H_0(R,*))=0$, then $KT(I,H_0(R,*))+P_{H_0(R,*)}KT(I,H_0(R,*))=0$ by (8), and (i) trivially holds.

Let us assume that $\mathrm{Heart}(H_0(R,*))\neq 0$. Let $\mathscr{I}$ be the set of nonzero $*$-ideals of R. By [7, 3.10, 3.13], for any $A\in\mathscr{I}$, $KT(A,H_0(R,*))$ is a nonzero semiideal of $H_0(R,*)$ contained in A. Thus $KT(A,H_0(R,*)) + P_{H_0(R,*)}KT(A,H_0(R,*))$ is a nonzero ideal of $H_0(R,*)$ contained in A, and

$$\mathrm{Heart}(H_0(R,*)) \subseteq \bigcap_{A\in\mathscr{I}} (KT(A,H_0(R,*)) + P_{H_0(R,*)}KT(A,H_0(R,*))) \subseteq \bigcap_{A\in\mathscr{I}} A = I,$$

which shows that $I\neq 0$. Now $KT(I,H_0(R,*)) + P_{H_0(R,*)}KT(I,H_0(R,*))$ is a nonzero ideal of $H_0(R,*)$ by [7, 3.10, 3.13], and (9) readily implies (ii).

The proof of (ii) applies to (iii) with obvious changes and the last assertion is an immediate consequence of (i), (ii) and (iii) with [7, 2.5, 3.5, 3.13]. □

4.8. Remark. $*$-Primeness is needed in (4.7)(ii)(iii) as shown by the following examples already used in [7, 3.4, 3.12]: Let R be the direct sum $S\boxplus\Phi$, where S is a $*$-simple associative triple system with involution $*$ such that $H(S,*)\neq 0$ and Φ is a field of characteristic not two. Extend the involution $*$ to R by $(x\boxplus\lambda)^*=x^*-\lambda$, for any $x\in S$, $\lambda\in\Phi$. Clearly R is semiprime and the $*$-heart of R is zero since $(S\boxplus 0)\cap(0\boxplus\Phi)=0$. But, since the characteristic is not two,

$$H_0(R,*) = H(R,*) = H(S,*)\boxplus 0 \cong H(S,*)$$

is simple by [7, 3.15(ii)], and $\mathrm{Heart}(H_0(R,*)) = H_0(R,*)$.

A similar example can be given for pairs by taking $R=S\boxplus V(\Phi)$, where S is a $*$-simple associative pair with $H(S,*)\neq 0$ and Φ is again a field of characteristic not two.

4.9. *Description of the heart of a Jordan system of Hermitian type:* Let J be a strongly prime Jordan algebra, triple or pair of hermitian type. In particular, J has a nonzero ideal of the form $H_0(R,*)$, where R is a $*$-prime associative system (cf. [2, 4.1, 5.3; 9, 4.1, 4.2, 4.3; 17, 15.2]). Let I be the $*$-heart of R. By (4.1), the heart of J coincides with the heart of $H_0(R,*)$. Thus, by (4.7),

(i) if J is an algebra, then $\mathrm{Heart}(J) = K(I,H_0(R,*))$ (cf. [7, 2.1]),

(ii) if J is a triple system, then $\mathrm{Heart}(J)$ is the ideal of $H_0(R,*)$ generated by $KT(I,H_0(R,*))$ (cf. [7, 3.9]), i.e.,

$$\mathrm{Heart}(J) = KT(I,H_0(R,*)) + P_{H_0(R,*)}KT(I,H_0(R,*)),$$

(iii) if J is a pair, then $\mathrm{Heart}(J)$ is the ideal of $H_0(R,*)$ generated by $KP(I,H_0(R,*))$ (cf. [7, 3.1]), i.e.,

$$\mathrm{Heart}(J)^\sigma = KP(I^\sigma,H_0(R,*)) + Q_{H_0(R^\sigma,*)}KP(I^{-\sigma},H_0(R,*)), \quad \sigma=\pm.$$

In all cases Heart(J) is an ample subspace $H_0(I,*)$ of I and

$$\mathrm{Heart}(J)=0 \quad\Leftrightarrow\quad *\text{-Heart}(R)=0.$$

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